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Computational Linear Algebra - MA 703
Problem Sheet 9

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1. Show that if $A \in \mathbb{R}^{m\times n}$ has rank p, then there exists an $X \in \mathbb{R}^{m\times p}$ and a $Y \in \mathbb{R}^{n\times p}$ such that $A = XY^T$, where rank(X)=rank(Y)=p.

2. Suppose $A(\alpha) \in \mathbb{R}^{m\times r}$ and $B(\alpha) \in \mathbb{R}^{r\times n}$ are matrices whose entries are differentiable functions of the scalar α. Show
$$\frac{d}{d\alpha}[A(\alpha)B(\alpha)] = \left[\frac{d}{d\alpha}A(\alpha)\right]B(\alpha) + A(\alpha)\left[\frac{d}{d\alpha}B(\alpha)\right].$$

3. Suppose $A(\alpha) \in \mathbb{R}^{n\times n}$ has entries that are differentiable functions of the scalar α. Assuming $A(\alpha)$ is always nonsingular, show
$$\left[\frac{d}{d\alpha}A(\alpha)^{-1}\right] = -A(\alpha)^{-1}\left[\frac{d}{d\alpha}A(\alpha)\right]A(\alpha)^{-1}.$$

4. Suppose $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$ and that $\phi(x) = \frac{1}{2}x^TAx - x^Tb$. Show that the gradient ϕ is given by $\nabla\phi(x) = \frac{1}{2}(A^T + A)x - b$.

5. Assume that both A and $A + uv^T$ are nonsingular where $A \in \mathbb{R}^{n\times n}$ and $u, v \in \mathbb{R}$. Show that if x solves $(A + uv^T)x = b$, then it also solves a perturbed right hand side problem of the form $Ax = b + \alpha u$. Give an expression for α in terms of A, u, and v.

6. Show that if $x \in \mathbb{R}^n$, then $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$.

7. Prove the Cauchy-Schwartz inequality by considering the inequality
$$0 \le (ax + by)^T(ax + by)$$
for suitable scalars a and b.

8. Verify that $\|\cdot\|_1, \|\cdot\|_2$, and $\|\cdot\|_\infty$ are vector norms.

9. For $x \in \mathbb{R}^n$ verify the following inequalities. When is equality achieved in each result?
   (a) $\|x\|_2 \le \|x\|_1 \le \sqrt{n}\|x\|_2$
   (b) $\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$
   (c) $\|x\|_\infty \le \|x\|_1 \le n\|x\|_\infty$.

10. Show that in $\mathbb{R}^n$, $x^{(i)} \to x$ if and only if $x_k^{(i)} \to x_k$ for $k = 1 : n$.

11. Show that any vector norm on $\mathbb{R}^n$ is uniformly continuous by verifying the inequality
$$\big|\|x\| - \|y\|\big| \le \|x - y\|.$$

12. Let $\|\cdot\|$ be a vector norm on $\mathbb{R}^m$ and assume $A \in \mathbb{R}^{m\times n}$. Show that if rank($A$)=$n$, then $\|x\|_A = \|Ax\|$ is a vector norm on $\mathbb{R}^n$.

13. Let x and y be in $\mathbb{R}^n$ and define $\psi : \mathbb{R} \to \mathbb{R}$ by $\psi(\alpha) = \|x - \alpha y\|_2$. Show that ψ is minimized when $\alpha = x^T y / y^T y$.

14. (a) Verify that $\|x\|_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$ is a vector norm on $\mathbb{C}^n$.
    (b) Show that if $x \in \mathbb{C}^n$ then $\|x\|_p \leq c(\|\mathrm{Re}(x)\|_p + \|\mathrm{Im}(x)\|_p)$.
    (c) Find a constant c_n such that $c_n(\|\mathrm{Re}(x)\|_2 + \|\mathrm{Im}(x)\|_2) \leq \|x\|_2$ for all $x \in \mathbb{C}^n$.

15. Prove or disprove:

$$v \in \mathbb{R}^n \Rightarrow \|v\|_1 \|v\|_\infty \leq \frac{1 + \sqrt{n}}{2} \|v\|_2.$$

16. Show $\|AB\|_p \leq \|A\|_p \|B\|_p$ where $1 \leq p \leq \infty$.

17. Let B be any submatrix of A. Show that $\|B\|_p \leq \|A\|_p$.

18. Show that if $D = \mathrm{diag}(\mu_1 \ldots, \mu_k) \in \mathbb{R}^{m \times n}$ with $k = \min\{m, n\}$, then $\|D\|_p = \max |\mu_i|$.

19. For $A \in \mathbb{R}^{m \times n}$ verify the following inequalities. Here $\|.\|_F$ refers the Frobenius norm.

    (a) $\|Ax\|_2 \leq \|A\|_F \leq \sqrt{n} \|A\|_2$
    (b) $\max_{i,j} |a_{ij}| \leq \|A\|_2 \leq \sqrt{mn} \;\; \max_{i,j} |a_{ij}|$
    (c) $\|A\|_1 = \max_{1 \leq j \leq n} \;\; \sum_{i=1}^{m} |a_{ij}|$
    (d) $\|A\|_\infty = \max_{1 \leq j \leq m} \;\; \sum_{j=1}^{n} |a_{ij}|$
    (e) $\frac{1}{\sqrt{n}} \|A\|_\infty \leq \|A\|_2 \leq \sqrt{m} \|A\|_\infty$
    (f) $\frac{1}{\sqrt{m}} \|A\|_1 \leq \|A\|_2 \leq \sqrt{n} \|A\|_1$.

20. Show that if $0 \neq s \in \mathbb{R}^n$ and $E \in \mathbb{R}^{n \times n}$, then

$$\left\| E \left(I - \frac{ss^T}{s^T s} \right) \right\|_F^2 = \|E\|_F^2 - \frac{\|E_s\|_2^2}{s^T s}.$$

21. Suppose $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Show that if $E = uv^T$ then $\|E\|_F = \|E\|_2 = \|u\|_2 \|v\|_2$ and that $\|E\|_\infty \leq \|u\|_\infty \|v\|_1$.

22. Suppose $A \in \mathbb{R}^{m \times n}, y \in \mathbb{R}^m$, and $0 \neq s \in \mathbb{R}^n$. Show that

$$E = (y - As)s^T / s^T s$$

has the smallest 2-norm of all $m \times n$ matrices E that satisfy $(A + E)s = y$.