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Concrete Mathematics - MA 201
Problem Sheet - 4

1. Find a fixed point (say, n_0) of $(101101101101011)_2$. Also find the smallest positive integer q such that $J^q((101101101101011)_2) = n_0$.

2. Say true or false with justification: $J(2n+1) - J(2n) = 2$ for any positive integer n.

3. For what values of n, is $J(n) = n/2$ true? Of course, here n is even.

4. For what values of m, is $2^m - 2$ a multiple of 3?

5. Write down the binary representation of n satisfying "$J(n) = n/2$" and conclude that cyclic-left-shifting (that is, $J(n)$) and one-place ordinary shifting (that is, halving n) are same.

6. Prove that the functions $A(n), B(n)$ and $C(n)$ of

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

where $A(n) = 2^m, B(n) = 2^m - 1 - \ell, C(n) = \ell,$ which solve

$$\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta, \quad \text{for } n \geq 1 \\ f(2n+1) &= 2f(n) + \gamma, \quad \text{for } n \geq 1. \end{aligned}$$

Here, as usual, $n = 2^m + \ell$ and $0 \leq \ell < 2^m$, for $n \geq 1$.

7. By induction on m, prove that $f(2^m + \ell) = 2^m$.

8. Find the values of parameters (α, β, γ), that will define $f(n) = n$.

9. Use the repertoire method to solve the general four-parameter recurrence

$$\begin{aligned} g(1) &= \alpha \\ g(2n+j) &= 3g(n) + \gamma n + \beta_j, \text{ for } j = 0, 1, \text{ and } n \geq 1. \end{aligned}$$

10. Use the repertoire method to solve the general five-parameter recurrence

$$\begin{aligned} h(1) &= \alpha \\ h(2n+j) &= 4g(n) + \gamma_j n + \beta_j, \text{ for } j = 0, 1, \text{ and } n \geq 1. \end{aligned}$$

11. For the original Josephus values $\alpha = 1, \beta = -1$ and $\gamma = 1$, find $J(100)$.
[Hint : $\beta_0 = \beta = -1$ and $\beta_1 = \gamma = 1$.]

12. Compute $f(19)$ from the recurrence, with initial conditions $f(1) = 34, f(2) = 5,$

$$\begin{aligned} f(3n) &= 10f(n) + 76 \text{ for } n \geq 1 \\ f(3n+1) &= 10f(n) - 2 \text{ for } n \geq 1 \\ f(3n+2) &= 10f(n) + 8, \text{ for } n \geq 1. \end{aligned}$$

13. Josephus had a friend who was saved by getting into the next-to-last position. What is $I(n)$, the number of the penultimate survivor when every second person is executed?

14. Suppose there are $2n$ people in a circle; the first n are "good guys" and the last n are "bad guys!" Show that there is always an integer m (depending on n) such that, if we go around the circle executing every mth person, all the bad guys are first to go. (For example, when $n = 3$ we can take $m = 5$; when $n = 4$ we can take $m = 30$.)

15. Suppose that Josephus finds himself in a given position j, but he has a chance to name the elimination parameter q such that every qth person is executed. Can he always save himself?

16. Generalizing the above exercise, let's say that a Josephus subset of $\{1, 2, \ldots, n\}$ is a set of k numbers such that, for some q, the people with the other $n - k$ numbers will be eliminated first. (These are the k positions of the "good guys" Josephus wants to save.) It turns out that when $n = 9$, three of the 29 possible subsets are non-Josephus, namely $\{1, 2, 5, 8, 9\}$, $\{2, 3, 4, 5, 8\}$, and $\{2, 5, 6, 7, 8\}$. There are 13 non-Josephus sets when $n = 12$, none for any other values of $n \leq 12$. Are non-Josephus subsets rare for large n?

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