Recurrence Relations and Their Solutions (Josephus Problem)

P. Sam Johnson

National Institute of Technology Karnataka (NITK)
Surathkal, Mangalore, India

The Josephus Problem : History

The problem is named after **Flavius Josephus**, a Jewish historian living in the 1st century. During Jewish-Roman war, Josephus and his 40 soldiers were trapped in a cave, the exit of which was blocked by Romans. They chose suicide over capture and decided that they would form a circle and start killing **every third remaining** person until two were left. But Josephus quickly calculated where he and his friend should stand in the circle, and finally, both of them escaped from execution.

A Roman portrait bust said to be of Josephus

(37AD-100AD)

Josephus Problem

There were 41 prisoners and every 3rd prisoner was being killed. Among them was a clever chap name Josephus who worked out the problem, stood at the surviving position, and lived on to tell the tale. Which number was he?

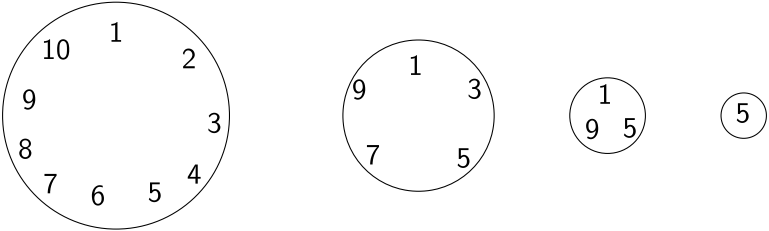
Josephus had a friend. The problem is to find the places of the two last remaining survivors. Where were they standing?

Josephus Problem (Generalized Case)

There are n people standing in a circle waiting to be executed. The counting out begins at some point in the circle and proceeds around the circle in a fixed direction. In each step, $k-1$ persons are skipped and the kth person is executed. The elimination proceeds around the circle (which is becoming smaller and smaller as the executed people are removed), until only $k-1$ persons remain, who are given freedom.

Josephus Problem (Simple Case $k = 2$)

The problem is restated simply for $k = 2$: there are n people standing in a circle, of which you are one. Someone outside the circle goes around clockwise and repeatedly eliminates every other person in the circle, until one person (the winner) remains. Where should you stand so you become the winner?



If $n = 10$, then the elimination order is $2, 4, 6, 8, 10, 3, 7, 1, 9$, so the winner (survivor's number) is 5.

Therefore, the survivor's number is 5 in a group of 10 people, it is denoted by $J(10) = 5$.

We look the solution at small cases.

n	1	2	3	4	5	6
$J(n)$	1	1	3	1	3	5

The **objective** is to find the survivor's number, $J(n)$, for a given positive integer n.

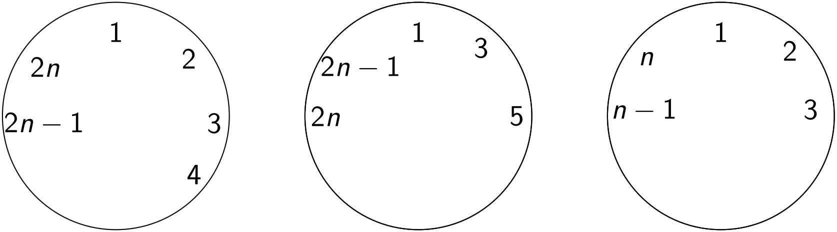
Can $J(n)$ be an even number ?

No. Because the first trip around the circle eliminates all the even numbers. Hence $J(n)$ must be an odd number. Note that we always assume that the entire process is started with the number 1.

What about the even case?

Suppose there are $2n$ people (even number).

After the first go-round we are left with only "n" odd numbers and 3 will be the next to go.



Beginning Stage   After First Round   Numbers Relabeled

This is just like starting out with n people, except that each person's number has been doubled and decreased by 1.

That is, $J(2n) = 2\ J(n) - 1$, for $n \geq 1$. Since $J(5) = 3$,

$$J(10) \;=\; 2\ J(5) - 1 = (2 \times 3) - 1 = 5.$$

Exercises

1. *Find* $J(20), J(40), J(80)$*. In general, find* $J(2^m \cdot 5)$*, for any positive integer* m*.*
2. *For any positive integer* m*, find* $J(2^m)$*. (OR) Can we say that the first person will survive whenever* n *is a power of* 2*?*
3. *Find the survivor's numbers of all even numbers upto* 16*.*

What about the odd case?

With $2n+1$ people, it turns out that person number 1 is wiped out just after person number $2n$, and we are left with $3, 5, 7, \ldots, 2n+1$.

Again we almost have the original situation with n people, but this time their numbers are doubled and increased by 1. Thus

$$J(2n+1) = 2\ J(n) + 1, \quad \text{for} \quad n \geq 1.$$

Combining these equations with $J(1) = 1$ gives us a recurrence that defined J in all cases:

$$\begin{aligned} J(1) &= 1 \\ J(2n) &= 2J(n) - 1, \quad \text{for} \quad n \geq 1 \\ J(2n+1) &= 2J(n) + 1, \quad \text{for} \quad n \geq 1. \end{aligned} \tag{1}$$

Exercise

4. *Find the survivor's numbers of all odd numbers upto* 15.

What is a closed form of (1)?

Exercise

5. *Using the recurrence relation (1), find* $J(42), J(39)$ *and* $J(61)$.

The above exercise will tell us that starting with any positive integer n, we can reach either $J(1)$ or $J(2)$ from the recurrence relation (1).
We may be able to spot a pattern and guess the recurrence solution if we write down $J(n)$ for small values on n.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9	11	13	15	1

We can group the values of $J(n)$ by powers of 2.

- In the table, the groups are separated by $\|$-lines.
- $J(n)$ is always 1 at the beginning of a group.
- $J(n)$ increases by 2 within a group.

Exercise

6. *Prove that any positive integer n can be written in the form $n = 2^m + \ell$ where 2^m is the largest power of 2 not exceeding n and where ℓ is remainder, $0 \leq \ell \leq 2^m - 1$.*

From the table, we guess that the solution $J(n)$ to (1) seems to be

$$J(2^m + \ell) = 2\ell + 1 \tag{2}$$

where 2^m is the largest power of 2 not exceeding n and where ℓ is remainder, $0 \leq \ell \leq 2^m - 1$.

Argument : Let $n = 2^m + \ell$, where $0 \leq \ell < 2^m$. We eliminate ℓ persons numbered $2, 4, \ldots, 2\ell$. We are left with 2^m persons. At this point, $(2\ell + 1)$ is the first person and he is the luck surviour.

Exercises

7. *Using induction, prove (2). Also find $J(102)$.*
   Note that the induction is on m.
8. *Find r such that $(121)_r = (144)_8$ where r and 8 are the bases.*

From the table and the recurrence solution (2), we observe that powers of 2 play an important role in our finding the solution. Hence we look at the base (radix) 2 representations of n and $J(n)$.

Suppose n's binary expansion is

$$n = (b_m b_{m-1} \cdots b_1 b_0)_2.$$

That is,

$$n = b_m 2^m + b_{m-1} 2^{m-1} + \cdots + b_1 2 + b_0,$$

where each b_i is either 0 or 1 where the leading bit b_m is 1.

Suppose $n = 2^m + \ell$, with the remainder ℓ satisfying $0 \le \ell \le 2^m - 1$. Then

$$\begin{aligned}
2^m &= (1 \quad 0 \quad \cdots \quad 0 \quad 0)_2 \\
\ell &= (0 b_{m-1} b_{m-2} \cdots b_1 b_0)_2 \\
2\ell &= (b_{m-1} b_{m-2} \cdots b_1 b_0 0)_2 \\
2\ell + 1 &= (b_{m-1} b_{m-2} \cdots b_1 b_0 1)_2 \\
n &= (1 b_{m-1} b_{m-2} \cdots b_1 b_0)_2.
\end{aligned}$$

From (2), we get

$$J\Big((1b_{m-1}b_{m-2}\cdots b_1b_0)_2\Big) = (b_{m-1}b_{m-2}\cdots b_1b_01)_2.$$

In the language (slang / lingo) of computer programming, we get $J(n)$ from n by doing a **one bit cyclic left-shift.**

Exercises

9. *Find* $J(343)$ *by using binary notation. Analyse the case when* $b_{m-1} = 0$.
10. *Is the following statement correct?*
If we start with n *and iterate the* J *function* $m+1$ *times* $\big($*applying* J *repeatedly with itself, we get* $J(n), J^2(n), J^3(n), \ldots, J^{m+1}(n)\big)$, *then we end up with* n *again. Note that each* n *is an* $(m+1)$*-bit number and we are doing* $m+1$ *one-bit cyclic left-shifts.*

Some Observations

- Since $J(n)$ is the survivor's number, $J(n)$ has to be "$\leq n$".
- If $J(n) < n$, then we can never get back up to n by continuing to iterate.
- Repeated application of J produces a sequence of decreasing values.
- When we do one-bit cyclic left shift, the "0" disappears when it becomes the leading bit, and the process can be continued.

Exercise

11. *Prove that for a given integer n, the following are equivalent:*
    (a) $J(n) = n$ *(thus,* $n = J(n) = J^2(n) = \cdots$*).*
    (b) *All bits of n are* 1.
    (c) $n = 2^{m+1} - 1$, *where* $n = 2^m + \ell, \quad 0 \leq \ell \leq 2^m - 1$.

Question-Answer

Question: Is there a positive integer n_0, for a given positive integer n such that

$$n \geq J(n) \geq J^2(n) \geq \cdots \geq n_0 = J(n_0) = J^2(n_0) = \cdots \quad ?$$

(OR) Does repeated application of J (to a given positive integer n) produce a sequence of numbers that eventually reach a "fixed point", say n_0?

If this is the case, how to find the fixed point?

Applying the function J repeatedly to the given positive integer n will produce a pattern of all 1's (all bits are 1, no zeros in between).

Suppose t is the number of 1 bits in the binary representation of n. Then the corresponding number, denoted by $v(n)$ is

$$v(n) = 1.2^{t-1} + 1.2^{t-2} + \cdots + 1.2 + 1 = 2^t - 1.$$

Thus $2^t - 1$ is a fixed point of J.

For example, $n = 13$ has the binary representation $(1101)_2$. Here $t = 3$.

Question-Answer

Question: Which powers of J has $2^3 - 1 = 7$ as a fixed point when we start with the number 13 ?

Answer: J^q, where q is the number of 1-bits (of the given number 7) appearing before the last 0-bit. Here q is any integer, greater than or equal to 2. That is, $J^2(13) = J^3(13) = J^4(13) = \cdots = 2^3 - 1 = 7$.

Exercises

12. *Find the smallest positive integer q such that $J^q((101101101101011)_2) = n_0$ for some n_0 (called* fixed point *of J^q).*
13. *Say true or false: $J(2n+1) - J(2n) = 2$ for any positive integer n.*
14. *For what values of n, is $J(n) = n/2$ true? Of course, here n is even.*
15. *For what values of m, is $2^m - 2$ a multiple of 3?*
16. *Write down the binary representation of n satisfying "$J(n) = n/2$" and conclude that cyclic-left-shifting (that is, $J(n)$) and one-place ordinary shifting (that is, halving n) are same.*

Closed Form for General Recurrence Relation

We now discuss to find a closed form for the more general recurrence relation:

$$\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta, \quad \text{for } n \geq 1 \\ f(2n+1) &= 2f(n) + \gamma, \quad \text{for } n \geq 1. \end{aligned} \tag{3}$$

We have the following table for the small values of n.

n	$f(n)$	Josephus Problem $\alpha = 1, \beta = -1, \gamma = 1$	
		Calculation	$J(n)$
1	α	1	1
2	$2\alpha + \beta$	$2 - 1$	1
3	$2\alpha + \gamma$	$2 + 1$	3

n	$f(n)$	Josephus Problem $\alpha = 1, \beta = -1, \gamma = 1$	
		Calculation	$J(n)$
4	$4\alpha + 3\beta$	$4 - 3$	1
5	$4\alpha + 2\beta + \gamma$	$4 - 2 + 1$	3
6	$4\alpha + \beta + 2\gamma$	$4 - 1 + 2$	5
7	$4\alpha \quad + 3\gamma$	$4 + 3$	7
8	$8\alpha + 7\beta$	$8 - 7$	1
9	$8\alpha + 6\beta + \gamma$	$8 - 6 + 1$	3
10	$8\alpha + 5\beta + 2\gamma$	$8 - 5 + 2$	5
$\vdots$	$\vdots$	$\vdots$	$\vdots$

From the table, we have the following observations:

- α's coefficient is n's largest power of 2 (say, 2^m)
- between powers of 2 (say, 2^m and 2^{m+1})
  - β's coefficient decreases by 1 (from $2^m - 1$ to 1 and finally 0)
  - γ's coefficient increases by 1 (initially 0 and increases from 1 to $2^m - 1$).

If we express the recurrence solution in the form

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma,$$

then from the observations, we have

$$\begin{aligned} A(n) &= 2^m \\ B(n) &= 2^m - 1 - \ell \\ C(n) &= \ell. \end{aligned}$$

Here, as usual, $n = 2^m + \ell$ and $0 \le \ell < 2^m$, for $n \ge 1$.

Recurrence Relation and Its Solution

Recurrence Relation

$$\begin{aligned} f(1) &= \alpha = 1 \\ f(2n) &= 2f(n) + \beta, \quad \text{for } n \geq 1 \\ f(2n+1) &= 2f(n) + \gamma, \quad \text{for } n \geq 1. \end{aligned} \tag{4}$$

Recurrence Solution

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma \tag{5}$$

where

$$\begin{aligned} A(n) &= 2^m \\ B(n) &= 2^m - 1 - \ell \\ C(n) &= \ell. \end{aligned}$$

Here, as usual, $n = 2^m + \ell$ and $0 \leq \ell < 2^m$, for $n \geq 1$.

Exercise

17. *Prove that the functions $A(n), B(n)$ and $C(n)$ of (5) which solve (4).*

Suppose $\alpha = 1, \beta = 0 = \gamma$. The solution to the recurrence relation

$$\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n), \quad \text{for } n \geq 1 \\ f(2n+1) &= 2f(n), \quad \text{for } n \geq 1, \end{aligned}$$

is $f(n) = 2^m$, where $n = 2^m + \ell, 0 \leq \ell < 2^m$, for $n \geq 1$.

Exercise

18. *By induction on m, prove that $f(2^m + \ell) = 2^m$.*

If we are given a solution $f(n)$, are there any constants (α, β, γ), that will define $f(n)$? Let us solve the following exercise.

Exercise

19. *Find the values of parameters (α, β, γ), that will define $f(n) = n$.*

Repertoire Method

The **repertoire method** is really a tool to help with the intuitive step of figuring out a closed form for a recurrence relation. It does so by breaking the original problem into smaller parts, with the hope they might be easier to solve.

We consider the generalized Josephus recurrence relation

$$\begin{aligned} f(1) &= \alpha \\ f(2n) &= 2f(n) + \beta, \quad \text{for } n \geq 1 \\ f(2n+1) &= 2f(n) + \gamma, \quad \text{for } n \geq 1. \end{aligned} \tag{6}$$

The idea of the **repertoire method** is to assume that there exists a solution of type

$$f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$$

where $A(n), B(n)$, and $C(n)$ are to be found.

Choosing particular values and combining them will give a solution.

We consider $\alpha = 1, \beta = \gamma = 0$. Then the recursion (6) becomes

$$\begin{aligned} A(1) &= 1 \\ A(2n) &= 2A(n), \quad \text{for } n \geq 1 \\ A(2n+1) &= 2A(n), \quad \text{for } n \geq 1. \end{aligned}$$

Hence $A(2^m + \ell) = 2^m$. By starting with a simple function, $f(n)$ and seeing if there are any constants (α, β, γ) that will define it.

Plugging the constant function $f(n) = 1$ into (6), we get $\alpha = 1, \beta = -1 = \gamma$ (after solving them). Hence

$$A(n) - B(n) - C(n) = 1.$$

Similiarly, we can plug in $f(n) = n$, we get $\alpha = 1, \beta = 0, \gamma = 1$. Hence

$$A(n) + C(n) = n.$$

The functions $A(n), B(n)$, and $C(n)$ of $f(n) = A(n)\alpha + B(n)\beta + C(n)\gamma$ in general, satisfy the equations

$$\begin{aligned} A(n) &= 2^m, \quad \text{where } n = 2^m + \ell \\ A(2n) &= 2A(n), \quad \text{for } n \geq 1 \\ A(2n+1) &= 2A(n), \quad \text{for } n \geq 1. \end{aligned}$$

Exercises

20. *Use the repertoire method to solve the general four-parameter recurrence*

$$\begin{aligned} g(1) &= \alpha \\ g(2n+j) &= 3g(n) + \gamma n + \beta, \ \textit{for } j = 0, 1, \ \textit{and } n \geq 1. \end{aligned}$$

21. *Use the repertoire method to solve the general five-parameter recurrence*

$$\begin{aligned} h(1) &= \alpha \\ h(2n+j) &= 4g(n) + \gamma_j n + \beta_j, \ \textit{for } j = 0, 1, \ \textit{and } n \geq 1. \end{aligned}$$

Generalized Josephus Recurrence Relation

We consider the generalized Josephus recurrence relation

$$\begin{aligned} f(1) &= \alpha \\ f(2n+j) &= 2f(n) + \beta_j, \quad \text{for } j = 0, 1 \text{ and } n \geq 1. \end{aligned}$$

Powers of 2 will play an important role in finding the solution, so we use the binary representations of n and $f(n)$.

$$\begin{aligned} f((b_m b_{m-1} b_{m-2} \cdots b_1 b_0)_2) &= 2f((b_m b_{m-1} b_{m-2} \cdots b_1)_2) + \beta_{b_0} \\ &= 4f((b_m b_{m-1} b_{m-2} \cdots b_2)_2) + 2\beta_{b_1} + \beta_{b_0} \\ &\quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\ &= 2^m f((b_m)_2) + 2^\beta b_{m-1} + \cdots + 2\beta_{b_1} + \beta_{b_0} \\ &= 2^m \alpha + 2^\beta b_{m-1} + \cdots + 2\beta_{b_1} + \beta_{b_0} \\ &= (\alpha \beta_{b_{m-1}} \cdots \beta_{b_1} \beta_{b_0})_2 \end{aligned}$$

Exercise

22. *For the original Josephus values $\alpha = 1, \beta = -1$ and $\gamma = 1$, find $J(100)$. [Hint : $\beta_0 = \beta = -1$ and $\beta_1 = \gamma = 1$.]*

Consider the recurrence relation

$$\begin{aligned} f(1) &= \alpha_1 \\ f(2) &= \alpha_2 \\ f(3n) &= 7f(n) + \beta_1 \\ f(3n+1) &= 7f(n) + \beta_2 \\ f(3n+2) &= 7f(n) + \beta_3, \quad \text{for} \quad n \geq 1. \end{aligned}$$

Powers of 3 play a role in our finding the solution : Start with the number "n" in base 3, and produce values $J_3(n)$ in base 7.

It has the following radix-changing solution :

$$f((b_m b_{m-1} b_{m-2} \cdots b_1 b_0)_3) = (\alpha \beta_{b_{m-1}} \cdots \beta_{b_1} \beta_{b_0})_7.$$

Consider the k-generalized Josephus problem

$$\begin{aligned} f(j) &= \alpha_j, \quad 1 \leq j \leq k-1 \\ f(kn+j) &= cf(n)+\beta_j, \quad 0 \leq j \leq k-1 \text{ and } n \geq 1. \end{aligned}$$

Powers of 3 play a role in our finding the solution : Start with the number "n" in base k, and produce values $J_k(n)$ in base c.

It has the following radix-changing solution :

$$f((b_m b_{m-1} b_{m-2} \cdots b_1 b_0)_k) = (\alpha \beta_{b_{m-1}} \cdots \beta_{b_1} \beta_{b_0})_c.$$

Exercise

23. *Compute $f(19)$ from the recurrence, with initial conditions $f(1) = 34, f(2) = 5$,*

$$\begin{aligned} f(3n) &= 10f(n)+76 \quad \textit{for} \quad n \geq 1 \\ f(3n+1) &= 10f(n)-2 \quad \textit{for} \quad n \geq 1 \\ f(3n+2) &= 10f(n)+8, \quad \textit{for} \quad n \geq 1. \end{aligned}$$

Exercises

24. *Josephus had a friend who was saved by getting into the next-to-last position. What is $I(n)$, the number of the penultimate survivor when every second person is executed?*
25. *Suppose there are $2n$ people in a circle; the first n are "good guys" and the last n are "bad guys!" Show that there is always an integer m (depending on n) such that, if we go around the circle executing every mth person, all the bad guys are first to go. (For example, when $n = 3$ we can take $m = 5$; when $n = 4$ we can take $m = 30$.)*
26. *Suppose that Josephus finds himself in a given position j, but he has a chance to name the elimination parameter q such that every qth person is executed. Can he always save himself?*

Exercise

27. *Generalizing the above exercise, let's say that a Josephus subset of* $\{1, 2, \ldots, n\}$ *is a set of* k *numbers such that, for some* q, *the people with the other* $n - k$ *numbers will be eliminated first. (These are the* k *positions of the "good guys" Josephus wants to save.) It turns out that when* $n = 9$, *three of the* 29 *possible subsets are non-Josephus, namely* $\{1, 2, 5, 8, 9\}$, $\{2, 3, 4, 5, 8\}$, *and* $\{2, 5, 6, 7, 8\}$. *There are* 13 *non-Josephus sets when* $n = 12$, *none for any other values of* $n \leq 12$. *Are non-Josephus subsets rare for large* n*?*

References

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